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CONTIGUITY RELATIONS FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. It is well known that the hypergeometric functions

$$ {}_2F_1(\alpha \pm 1, \beta, \gamma; t), \quad {}_2F_1(\alpha, \beta \pm 1, \gamma; t), \quad {}_2F_1(\alpha, \beta, \gamma \pm 1; t), $$

which are contiguous to ${}_2F_1(\alpha, \beta, \gamma; t)$, can be expressed in terms of

$$ {}_2F_1(\alpha, \beta, \gamma; t) \quad \text{and} \quad {}_2F_1'(\alpha, \beta, \gamma; t). $$

We explain how to derive analogous formulas for generalized hypergeometric functions. Our main point is that such relations can be deduced from the geometry of the cone associated in a recent paper by B. Dwork and F. Loeser to a generalized hypergeometric series.

1. INTRODUCTION

Let $A = (A_{ij})$ be an $(m \times n)$-matrix with entries in $\mathbb{Z}$. For $i = 1, \ldots, m$, let ℓ_i be the linear form defined by the ith row of A:

$$ \ell_i(s_1, \ldots, s_n) = \sum_{j=1}^{n} A_{ij} s_j. $$

Let $a = (a_1, \ldots, a_m) \in \mathbb{C}^m$. We suppose a satisfies the condition:

$$ \text{If } a_i \in \mathbb{N}^\times, \text{ then } A_{ij} \in \mathbb{N} \text{ for } j = 1, \ldots, n, \tag{1.1} $$

where $\mathbb{N}$ denotes the nonnegative integers and $\mathbb{N}^\times$ denotes the positive integers. We may then define the generalized hypergeometric series

$$ Y(a; t) = \sum_{s \in \mathbb{N}^n} t_1^{s_1} \cdots t_n^{s_n} \frac{(-1)^{s_1 + \cdots + s_n}}{s_1! \cdots s_n!} \prod_{i=1}^{m} (a_i)_{\ell_i(s)}, $$

where as usual for $\rho \in \mathbb{Z}$, $(z)_\rho = \Gamma(z + \rho)/\Gamma(z)$.

Let ϵ_i be the unit vector in the ith coordinate direction in $\mathbb{C}^m$. It is easy to verify that if a, $a + \epsilon_i$ satisfy (1.1), then

$$ a_i Y(a + \epsilon_i; t) = (a_i + \ell_i(\delta)) Y(a; t), \tag{1.2} $$

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where $\ell_i(\delta) = \sum_{j=1}^n A_{ij}\delta_j$ and $\delta_j = t_j\partial/\partial t_j$. The purpose of this note is to invert this relation. We solve the following problem.

Problem. Find $P_i \in \mathbb{Q}(a)[t, \partial/\partial t_1, \dots, \partial/\partial t_n]$ such that for generic values of a,

$$Y(a-\epsilon_i; t) = P_i(a, t, \partial/\partial t_1, \dots, \partial/\partial t_n)Y(a; t). \tag{1.3}$$

We give an algorithm for constructing P_i and show that the coefficients in $\mathbb{Q}(a)$ appearing in P_i have denominators which are products of linear factors involving the faces of codimension one of the cone associated with Y in earlier work [1, 2, 3]. We give estimates for the degree of P_i as a polynomial in t and we describe the set of $a \in \mathbb{C}^m$ for which (1.3) is valid. Under an additional condition, which is satisfied by all the classically studied hypergeometric series, we bound the order of P_i as a partial differential operator.

This problem has a lengthy history. The function ${}_2F_1$ had been treated by Gauss and Appell's F_1 had been treated by Lavasseur in his 1893 Paris thesis. Professor Kita has brought to our attention the recent works [5, 7]. The methods and scope of [5, 7] are quite different.

2. Exponential modules

Let $A^{(j)}$ be the jth column of the matrix A. We recall that, in earlier work, the polynomial

$$-g(t, x) = x_1 + \cdots + x_m + \sum_{j=1}^n t_j x^{A^{(j)}},$$

where $x^{A^{(j)}} = x_1^{A_{1j}} \cdots x_m^{A_{mj}}$, has been associated with $Y(a; t)$. Let $\Omega = \mathbb{Q}(a)$, $R' = \Omega(t)[x_1, x_1^{-1}, \dots, x_m, x_m^{-1}]$, $E_i = x_i\partial/\partial x_i$ for $i = 1, \dots, m$, and $g_i = E_i(g)$. Define operators on R': $D_{a,i,t} = E_i + a_i + g_i$ for $i = 1, \dots, m$, and $\sigma_j = \partial/\partial t_j - x^{A^{(j)}} (= \partial/\partial t_j + \partial g/\partial t_j)$ for $j = 1, \dots, n$. The $D_{a,i,t}$ and σ_j commute with one another for all i and j. We define

$$\mathscr{W}'_{a,t} = R' / \sum_{i=1}^m D_{a,i,t} R',$$

which is viewed as an $\mathscr{R}_1$-module, where $\mathscr{R}_1$ is the noncommutative ring $\Omega(t)[\sigma_1, \dots, \sigma_n]$.

We make the hypothesis

$$a_i \notin \mathbb{N}^\times \quad \text{for } i = 1, \dots, m. \tag{2.1}$$

Let

$$R^* = \left\{ \sum_{u\in\mathbb{Z}^m} A_u(t)x^{-u} \mid A_u(t) \in \Omega[[t]] \right\}$$

and let $\xi^*_{a,t} \in R^*$ be defined by

$$\xi^*_{a,t} = \exp\left(-\sum_{j=1}^n t_j x^{A^{(j)}}\right) \cdot \sum_{u\in\mathbb{Z}^m} \frac{\prod_{i=1}^m (a_i)_{u_i}}{x^u}. \tag{2.2}$$

By a direct calculation, for $u \in \mathbb{Z}^m$

$$\left(\prod_{i=1}^{m}(a_i)_{u_i}\right)Y(a+u;t) = \langle \xi^*_{a,t}, x^u\rangle, \tag{2.3}$$

where for $\xi^* \in R^*$, $\xi \in R'$, $\langle \xi^*, \xi\rangle$ is defined to be the coefficient of x^0 in the product $\xi^*\xi$. In particular, taking $u = 0$ gives

$$Y(a;t) = \langle \xi^*_{a,t}, 1\rangle.$$

For any $\xi^* \in R^*$, $\xi \in R'$, one checks easily that

$$\frac{\partial}{\partial t_j}\langle \xi^*, \xi\rangle = \langle \sigma_j^*(\xi^*), \xi\rangle + \langle \xi^*, \sigma_j(\xi)\rangle,$$

where $\sigma_j^* = \partial/\partial t_j + x^{A^{(j)}}$. From (2.2) it follows that $\sigma_j^*(\xi^*_{a,t}) = 0$, hence

$$\frac{\partial}{\partial t_j}\langle \xi^*_{a,t}, \xi\rangle = \langle \xi^*_{a,t}, \sigma_j(\xi)\rangle. \tag{2.4}$$

Applying this with $\xi = 1$, we conclude that for $P \in \mathbb{Q}(a)[t, Z_1, \dots, Z_n]$,

$$\begin{aligned} P(a, t, \partial/\partial t_1, \dots, \partial/\partial t_n)Y(a;t) &= P(a, t, \partial/\partial t_1, \dots, \partial/\partial t_n)\langle \xi^*_{a,t}, 1\rangle \\ &= \langle \xi^*_{a,t}, P(a, t, \sigma_1, \dots, \sigma_n)1\rangle. \end{aligned} \tag{2.5}$$

Under the pairing $\langle\ ,\ \rangle$, the adjoint on R^* of the mapping $D_{a,i,t}$ on R' is the mapping $D^*_{a,i,t} = -E_i + a_i + g_i$. One checks that $D^*_{a,i,t}(\xi^*_{a,t}) = 0$ for $i = 1, \dots, m$. It follows that $\xi^*_{a,t}$ annihilates $\sum_{i=1}^m D_{a,i,t}R'$ under the pairing. Taking $u = -\epsilon_i$ in (2.3) and comparing with (2.5) reduces the problem stated in the introduction to the problem of finding $P_i \in \mathscr{R}_1$ such that

$$\frac{a_i - 1}{x_i} \equiv P_i(a, t, \sigma_1, \dots, \sigma_n)1 \pmod{\sum_{i=1}^m D_{a,i,t}R'}. \tag{2.6}$$

One then has

$$P_i(a, t, \partial/\partial t_1, \dots, \partial/\partial t_n)Y(a;t) = Y(a-\epsilon_i;t). \tag{2.7}$$

Let $\ell_i(t\sigma) = \sum_{j=1}^m A_{ij}t_j\sigma_j$. One checks from the definitions that

$$a_i + \ell_i(t\sigma) - \ell_i(\delta) = -E_i + x_i + D_{a,i,t},$$

hence for $v \in \mathbb{Z}^m$,

$$(a_i + \ell_i(t\sigma) + v_i)x^v \equiv x^{v+\epsilon_i} \pmod{\sum_{i=1}^m D_{a,i,t}R'}.$$

One then proves by induction that for $r \in \mathbb{N}^m$,

$$x^r \equiv \prod_{i=1}^m (a_i + \ell_i(t\sigma))_{r_i}1 \pmod{\sum_{i=1}^m D_{a,i,t}R'}. \tag{2.8}$$

Let $\tilde{H}_0$ be the monoid generated by $\epsilon_1, \ldots, \epsilon_m, A^{(1)}, \ldots, A^{(n)}$. If $u \in \tilde{H}_0$, then $u = r + \sum_{j=1}^{n} s_j A^{(j)}$, where $r \in \mathbb{N}^m$, $(s_1, \ldots, s_n) \in \mathbb{N}^n$. One has trivially

$$(-\sigma_1)^{s_1} \cdots (-\sigma_n)^{s_n}(x^r) = x^{r+\sum_{j=1}^{n} s_j A^{(j)}} = x^u.$$

Hence by (2.8),

$$x^u \equiv (-\sigma_1)^{s_1} \cdots (-\sigma_n)^{s_n} \prod_{i=1}^{m} (a_i + \ell_i(t\sigma))_{r_i} 1 \pmod{\sum_{i=1}^{m} D_{a,i,t} R'}. \tag{2.9}$$

Thus to find P_i satisfying (2.6), it suffices to find a formula of the type

$$\frac{a_i - 1}{x_i} \equiv \sum_{u \in \tilde{H}_0} c_{i,u} x^u \pmod{\sum_{i=1}^{m} D_{a,i,t} R'}, \tag{2.10}$$

where the sum on the right-hand side is finite and each $c_{i,u}$ lies in $\mathbb{Q}(a)[t]$. For future use, we note that (2.9) combined with (2.5) gives for $u \in \tilde{H}_0$

$$\langle \xi^*_{a,t}, x^u \rangle = (-1)^{s_1+\cdots+s_n} \prod_{j=1}^{n} \left(\frac{\partial}{\partial t_j}\right)^{s_j} \prod_{i=1}^{m} (a_i + \ell_i(\delta))_{r_i} Y(a; t). \tag{2.11}$$

3. The contiguity algorithm

Let $\mathscr{C}$ be the cone in $\mathbb{R}^m$ determined by the monomials of g:

$$\mathscr{C} = \{ z \in \mathbb{R}^m \mid z = \sum_{i=1}^{m} r_i \epsilon_i + \sum_{j=1}^{n} s_j A^{(j)}, \text{ all } r_i, s_j \in [0, \infty) \}.$$

Let $\hat{H}_0 = \mathscr{C} \cap \mathbb{Z}^m$. We introduce $\hat{H}_0$ because it can be characterized by a system of linear inequalities.

Lemma 3.1. *There exists $w \in \tilde{H}_0$ such that $\hat{H}_0 + w \subseteq \tilde{H}_0$. In particular, we may take $w = \sum_{i=1}^{m} T_i \epsilon_i$, where*

$$T_i = \sup\left(0, -1 + \sum_{j=1}^{n} \sup(0, -A_{ij})\right).$$

Remark. For classical hypergeometric functions the matrices A are made explicit in the appendix of [2] and it is not hard to check that in all these cases we have $\tilde{H}_0 = \hat{H}_0$, i.e., one may take $w = 0$ in the classical examples.

Proof. If $v \in \hat{H}_0$ then $v = \sum_{i=1}^{m} r_i \epsilon_i + \sum_{j=1}^{n} s_j A^{(j)}$ where all r_i and s_j are nonnegative. Putting $r_i = \alpha_i + \alpha_i'$, $\alpha_i \in \mathbb{N}$, $\alpha_i' \in [0, 1)$, and $s_j = \beta_j + \beta_j'$, $\beta_j \in \mathbb{N}$, $\beta_j' \in [0, 1)$, we conclude that $v = u + \mu$, where $u \in \tilde{H}_0$ and $\mu = \sum_{i=1}^{m} \alpha_i' \epsilon_i + \sum_{j=1}^{n} \beta_j' A^{(j)} \in \hat{H}_0$. Since μ lies in a bounded set, there are only a finite number of possibilities for μ and hence there exists $w \in \mathbb{N}^m$ such that $w + \mu \in \mathbb{N}^m \subseteq \tilde{H}_0$ for all μ. This shows the existence of w. To check our particular choice for w it is enough to check that for all i, $\alpha_i' + \sum_{j=1}^{n} \beta_j' A_{ij} \in \mathbb{Z}$ implies that $\alpha_i' + \sum_{j=1}^{n} \beta_j' A_{ij} + T_i \geq 0$. This follows from the fact that if

$\inf_{j=1,\dots,n}\{A_{ij}\} < 0$, then

$$\sum_{j=1}^{n} \beta'_j A_{ij} > \sum_{j=1}^{n} \inf(0, A_{ij}) = -1 - T_i. \quad \square$$

We now recall that the cone $\mathscr{C}$ may also be defined by linear inequalities. Let $\tau_1, \dots, \tau_\rho$ be the hyperplanes through the faces of $\mathscr{C}$ of codimension one. Then for $k = 1, \dots, \rho$, τ_k is defined by a linear form

$$f_k(u) = \sum_{i=1}^{m} B_{ki} u_i,$$

where the B_{ki} are integers with greatest common divisor 1, $f_k(u) = 0$ is the equation of τ_k, and $f_k(u) \geq 0$ for all $u \in \mathscr{C}$. Let us write

$$\begin{aligned} f_k(D_a) &= \sum_{i=1}^{m} B_{ki} D_{a,i,t} \\ f_k(g) &= \sum_{i=1}^{m} B_{ki} g_i \\ &= -\sum_{i=1}^{m} x_i f_k(\epsilon_i) - \sum_{j=1}^{n} t_j x^{A^{(j)}} f_k(A^{(j)}) \\ f_k(E) &= \sum_{i=1}^{m} B_{ki} E_i. \end{aligned}$$

The key point is that all monomials appearing in $f_k(g)$ have exponents lying in the region $f_k(u) \geq 1$.

Lemma 3.2. *For each $v \in \mathbb{Z}^m$ there exists a representation*

$$x^v \equiv \sum_{u \in \tilde{H}_0} c_{v,u} x^u \pmod{\sum_{i=1}^{m} D_{a,i,t} R'},$$

where the sum on the right-hand side is finite and each $c_{v,u} \in \mathbb{Q}(a)[t]$.

Proof. We use induction on $N_v = \sum_{k=1}^{\rho} \sup(0, f_k(w - v))$. If $N_v = 0$, then $f_k(v - w) \geq 0$ for all k which shows that $v \in w + \hat{H}_0 \subseteq \tilde{H}_0$ by Lemma 3.1. Thus we may assume $f_k(v - w) < 0$ for some k. We compute

$$f_k(D_a) x^v = (f_k(a) + f_k(v)) x^v + f_k(g) x^v,$$

and so

$$x^v \equiv -\frac{1}{f_k(a) + f_k(v)} f_k(g) x^v \pmod{\sum_{i=1}^{m} D_{a,i,t} R'}. \tag{3.3}$$

We now apply the induction hypothesis to $f_k(g)x^v$, which is a $\mathbb{Z}[t]$-linear combination of terms $x^{v'}$ such that $N_{v'} \leq N_v - 1$. $\square$

Note that equation (3.3) remains valid under specialization of a for all a such that $f_k(a)+f_k(v)\neq 0$. As an immediate consequence of the proof, we have:

Corollary 3.4.

$$\deg_t c_{v,u} \le N_v, \tag{3.5}$$

$$\left(\prod_{k=1}^{\rho}(f_k(a)+f_k(v))_{\sup(0,f_k(w-v))}\right)c_{v,u}\in \mathbb{Q}[a,t]. \tag{3.6}$$

Specializing v to $-\epsilon_i$ and combining Lemma 3.2 with (2.11), (2.3) and Corollary 3.4 gives:

Theorem 3.7. *There exists* $P_i \in \mathbb{Q}(a)[t, \partial/\partial t_1, \dots, \partial/\partial t_n, \delta_1, \dots, \delta_n]$ *such that*

$$Y(a-\epsilon_i;t) = P_i(a,t,\partial/\partial t_1,\dots,\partial/\partial t_n,\delta_1,\dots,\delta_n)Y(a;t).$$

As polynomials in t, *the coefficients of* P_i *have degree bounded by*

$$N_{-\epsilon_i} = \sum_{k=1}^{\rho}\sup(0, f_k(w+\epsilon_i))$$

and $H_i(a)P_i$ *has coefficients in* $\mathbb{Q}[a,t]$, *where*

$$H_i(a) = \prod_{i=1}^{\rho}(f_k(a)+f_k(-\epsilon_i))_{\sup(0,f_k(w+\epsilon_i))}.$$

Thus this contiguity relation is valid provided $a_j \notin \mathbb{N}^{\times}$ *for* $j=1,\dots,m$ *and* $H_i(a)\neq 0$.

Of course, if one expresses P_i as a polynomial in the $\partial/\partial t_i$'s only (i.e., replace δ_i by $t_i\partial/\partial t_i$), then the degrees of its coefficients as polynomials in t change. These new degrees can be bounded by the methods of the next section, under the additional assumption that $\hat{H}_0 = \tilde{H}_0$.

We believe that this theorem gives the basic set of contiguity relations. We observe that other contiguity relations may be deduced from

$$\langle \xi^*_{a,t}, D_{a,i,t}x^v\rangle = 0$$

for all $v\in\mathbb{Z}^m$, $i=1,\dots,m$, together with either (2.11) for $v\in\tilde{H}_0$ or (2.3) for arbitrary $v\in\mathbb{Z}^m$.

4. Bounding the order of P_i

To bound the order of P_i as a differential operator, we introduce some auxiliary functions. For $u\in\tilde{H}_0$, put

(4.1)

$$W(u) = \inf\left\{\sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j \mid u = \sum_{i=1}^{m} r_i\epsilon_i + \sum_{j=1}^{n} s_jA^{(j)},\ r_i, s_j\in\mathbb{N} \text{ for all } i,j\right\}.$$

From (2.11) we see that the differential operator on $Y(a;t)$ that corresponds to x^u (more precisely, that corresponds to a representation of u minimizing $\sum_{i=1}^m r_i + \sum_{j=1}^n s_j$) has order $W(u)$. Thus the problem of bounding the order of the differential operator corresponding to x^v, $v \in \mathbb{Z}^m$, is reduced to the problem of bounding $W(u)$ as u ranges over all terms with $c_{v,u} \neq 0$ on the right-hand side of Lemma 3.2. To accomplish this, we need to extend the definition of W to all $u \in \mathbb{Z}^m$.

It is clear from (4.1) that if $u_1, u_2 \in \tilde{H}_0$, then

$$W(u_1 + u_2) \leq W(u_1) + W(u_2). \tag{4.2}$$

Any $u \in \mathbb{Z}^m$ can be written $u = u_1 - u_2$ with $u_1, u_2 \in \tilde{H}_0$. If in addition $u \in \tilde{H}_0$, then (4.2) implies $W(u_1) - W(u_2) \leq W(u)$. Thus we may extend (4.1) by defining for $u \in \mathbb{Z}^m$

$$W(u) = \sup\{W(u_1) - W(u_2) \mid u = u_1 - u_2,\ u_1, u_2 \in \tilde{H}_0\}. \tag{4.3}$$

Remark. We shall establish later that, under the hypothesis $\hat{H}_0 = \tilde{H}_0$, $W(u) < \infty$ for all $u \in \mathbb{Z}^m$. This will show that our bound on the order of P_i is nontrivial.

Lemma 4.4. *If* $u \in \mathbb{Z}^m$, $u' \in \tilde{H}_0$, *then*

$$W(u + u') \leq W(u) + W(u').$$

Proof. Pick $u_1, u_2 \in \tilde{H}_0$ such that $u + u' = u_1 - u_2$. By (4.2), $W(u' + u_2) \leq W(u') + W(u_2)$, hence

$$W(u_1) - W(u_2) \leq W(u_1) + W(u') - W(u' + u_2).$$

But $u_1 - (u' + u_2) = u$, so $W(u_1) - W(u' + u_2) \leq W(u)$. This implies the lemma. □

Proposition 4.5. *For* $v \in \mathbb{Z}^m$, *the partial differential operator corresponding to* x^v *under* (2.11) *and Lemma* 3.2 *has order* $\leq W(v) + N_v$.

Proof. The proof is by induction on N_v. If $N_v = 0$, then as noted in the proof of Lemma 3.2, $v \in \tilde{H}_0$. Suppose $N_v > 0$. By Lemma 4.4, $f_k(g)x^v$ is a $\mathbb{Z}[t]$-linear combination of terms $x^{v'}$ such that $W(v') \leq W(v) + 1$. Since $N_{v'} \leq N_v - 1$, we are done by (3.3) and the induction hypothesis. □

Corollary 4.6. *The partial differential operator* P_i *of Theorem* 3.7 *can be chosen to have order* $\leq W(-\epsilon_i) + N_{-\epsilon_i}$.

We now show this bound is nontrivial when $\hat{H}_0 = \tilde{H}_0$. We introduce a function W' on $\hat{H}_0$ defined by

$$W'(u) = \inf\left\{ \sum_{i=1}^m r_i + \sum_{j=1}^n s_j \,\middle|\, u = \sum_{i=1}^m r_i\epsilon_i + \sum_{j=1}^n s_j A^{(j)},\ r_i, s_j \in [0, \infty) \text{ for all } i, j \right\}. \tag{4.7}$$

Trivially, $W'(u) \leq W(u)$ for all $u \in \tilde{H}_0$. The function W' has a geometric interpretation: $W'(u)$ is the smallest nonnegative real number such that $u \in$

$W'(u)\Delta$, where Δ is the convex hull of the points $\epsilon_1, \dots, \epsilon_m, A^{(1)}, \dots, A^{(n)}$ and the origin and $W'(u)\Delta$ is the dilation of Δ by the factor $W'(u)$. Let $\lambda_1, \dots, \lambda_p$ be linear forms defining the codimension-one faces of Δ that do not contain the origin. We assume them to be normalized so that the corresponding codimension-one face lies in the hyperplane $\lambda_i(u) = 1$ for $i = 1, \dots, p$. This determines the λ_i's uniquely. Then for $u \in \hat{H}_0$,

$$W'(u) = \sup\{\lambda_i(u) \mid i = 1, \dots, p\}. \tag{4.8}$$

Lemma 4.9. *Fix $u \in \hat{H}_0$. Then the set $\{W'(u+u_1) - W'(u_1) \mid u_1 \in \hat{H}_0\}$ is bounded above and below.*

Proof. From the definition of W' it is clear that

$$W'(u+u_1) \leq W'(u) + W'(u_1),$$

hence the given set is bounded above by $W'(u)$. By (4.8) we may choose $i_1, i_2 \in \{1, \dots, p\}$ such that

$$W'(u+u_1) = \lambda_{i_1}(u+u_1), \quad W'(u_1) = \lambda_{i_2}(u_1).$$

Then $(\lambda_{i_1} - \lambda_{i_2})(u_1) \leq 0$ but $(\lambda_{i_1} - \lambda_{i_2})(u+u_1) \geq 0$, hence there exists $\alpha \in [0, 1]$ such that

$$(\lambda_{i_1} - \lambda_{i_2})(\alpha u + u_1) = 0. \tag{4.10}$$

Then

$$\begin{aligned} W'(u+u_1) - W'(u_1) &= \lambda_{i_1}(u+u_1) - \lambda_{i_2}(u_1) \\ &= (1-\alpha)\lambda_{i_1}(u) + \alpha\lambda_{i_2}(u) \end{aligned}$$

by (4.10). This latter quantity is clearly bounded above and below independently of u_1. □

Lemma 4.11. *Suppose $\hat{H}_0 = \tilde{H}_0$. There exists a positive constant κ such that for all $u \in \tilde{H}_0$,*

$$W(u) \leq W'(u) + \kappa.$$

Proof. Let $u \in \tilde{H}_0$. Choose $r_i, s_j \in [0, \infty)$ such that

$$u = \sum_{i=1}^{m} r_i\epsilon_i + \sum_{j=1}^{n} s_j A^{(j)} \quad \text{and} \quad W'(u) = \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j.$$

Put $[u] = \sum_{i=1}^m [r_i]\epsilon_i + \sum_{j=1}^n [s_j]A^{(j)} \in \tilde{H}_0$. Then $u - [u] = \mu \in \mathbb{Z}^m \cap \mathscr{C} = \tilde{H}_0$. Furthermore, μ lies in a bounded (hence finite) subset of $\tilde{H}_0$. Let κ be the maximum value of W on this finite subset. Now $u = [u] + \mu$, hence

$$\begin{aligned} W(u) &\leq \sum_{i=1}^{m}[r_i] + \sum_{j=1}^{n}[s_j] + \kappa \\ &\leq W'(u) + \kappa. \quad \square \end{aligned}$$

Proposition 4.12. *Suppose $\hat{H}_0 = \tilde{H}_0$. Then $W(u) < \infty$ for all $u \in \mathbb{Z}^m$.*

Proof. Write $u = u_1 - u_2$ with $u_1, u_2 \in \tilde{H}_0$. By Lemma 4.4, $W(u) \le W(u_1) + W(-u_2)$. Thus it suffices to show $W(-u) < \infty$ for all $u \in \tilde{H}_0$. So suppose $-u = u_1 - u_2$ with $u, u_1, u_2 \in \tilde{H}_0$. Then $u_2 = u_1 + u$, so

$$\begin{aligned} W(u_1) - W(u_2) &= W(u_1) - W(u_1 + u) \\ &\le W'(u_1) + \kappa - W'(u_1 + u) \end{aligned}$$

by Lemma 4.11. By Lemma 4.9, this quantity is bounded above independently of u_1. □

5. Examples

Consider the classical Gaussian hypergeometric function

$${}_2F_1(\alpha, \beta, \gamma; t) = \sum_{s=0}^{\infty} \frac{(\alpha)_s(\beta)_s}{(\gamma)_s s!} t^s.$$

Using the relation $(\gamma)_s(1-\gamma)_{-s} = (-1)^s$ we have

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma; t) &= \sum_{s=0}^{\infty} (\alpha)_s(\beta)_s(1-\gamma)_{-s} \frac{(-t)^s}{s!} \\ &= Y(\alpha, \beta, 1-\gamma; t), \end{aligned}$$

so $a = (a_1, a_2, a_3) = (\alpha, \beta, 1-\gamma)$ and $\ell_1(s) = \ell_2(s) = s$, $\ell_3(s) = -s$. This corresponds to

$$-g(t, x) = x_1 + x_2 + x_3 + t\frac{x_1 x_2}{x_3}.$$

The codimension-one faces of the corresponding cone $\mathscr{C}$ are given by the forms $f_1(u) = u_2 + u_3$, $f_2(u) = u_1 + u_3$, $f_3(u) = u_1$, $f_4(u) = u_2$ and one checks that $\hat{H}_0 = \tilde{H}_0$, hence $w = 0$ in Lemma 3.1. One has $f_1(-\epsilon_1), f_4(-\epsilon_1) \ge 0$ but $f_2(-\epsilon_1) = f_3(-\epsilon_1) = -1$. Following the algorithm described in the proof of Lemma 3.2 by first applying $f_2(D_a)$ to $1/x_1$ and then applying $f_3(D_a)$ to x_3/x_1 gives

$$\frac{\alpha - 1}{x_1} \equiv \frac{\alpha - 1 + x_3 + tx_2}{\alpha - \gamma} \quad \left(\text{mod} \sum_{i=1}^{3} D_{a,i,t}R'\right).$$

By (2.9),

$$x_3 \equiv (1 - \gamma - t\sigma)1 \ \left(\text{mod} \sum_{i=1}^{3} D_{a,i,t}R'\right),$$

$$x_2 \equiv (\beta + t\sigma)1 \ \left(\text{mod} \sum_{i=1}^{3} D_{a,i,t}R'\right),$$

so

$$\frac{\alpha - 1}{x_1} \equiv \frac{(\alpha - \gamma - t\sigma) + t(\beta + t\sigma)}{\alpha - \gamma} 1 \quad \left(\text{mod} \sum_{i=1}^{3} D_{a,i,t}R'\right).$$

From (2.7) we get

$$(\gamma-\alpha)\,{}_2F_1(\alpha-1\,,\,\beta\,,\,\gamma\,;\,t)=\left(t(1-t)\frac{\partial}{\partial t}+(\gamma-\alpha-\beta t)\right){}_2F_1(\alpha\,,\,\beta\,,\,\gamma\,;\,t)\,,$$

a well-known classical formula (see [4, section 2.8, equation (23)] or [6, Chapter VI, section 24]).

We give some details for the calculation of the contiguity relations for the Lauricella series F_A, which we write in the form

$$Y(a\,;\,t)=\sum_{s\in\mathbb{N}^n}c(s)\frac{t^s}{s_1!\cdots s_n!}\,,$$

where

$$c(s)=(a_{2n+1})_{s_1+\cdots+s_n}\frac{\prod_{i=1}^n(a_{i+n})_{s_i}}{\prod_{i=1}^n(1-a_i)_{s_i}}.$$

The associated polynomial [2] is

$$-g=x_1+\cdots+x_{2n+1}+\sum_{j=1}^n t_jx_{2n+1}\frac{x_{n+j}}{x_j}.$$

There are 2^n+2n linear forms which define the associated cone:

$$f_j(u)=u_{n+j}\qquad(j=1\,,\,\ldots\,,\,n)\,,$$
$$f_{n+j}(u)=u_{n+j}+u_j\qquad(j=1\,,\,\ldots\,,\,n)\,,$$

and for each subset S of $\{1\,,\,\ldots\,,\,n\}$ the form

$$f_S(u)=u_{2n+1}+\sum_{j\in S}u_j.$$

Clearly these forms are nonnegative on the cone of g. We omit the proof that these forms define the cone.

We give some of the calculations for $Y(a-\epsilon_1\,;\,t)$. By applying $f_{n+1}(D_a)$ to $1/x_1$ we obtain

$$(a_{n+1}+a_1-1)\frac{1}{x_1}\equiv(x_1+x_{n+1})\frac{1}{x_1}=1+\frac{x_{n+1}}{x_1}.$$

Letting $S=\{1\}$ and applying $f_S(D_a)$ to x_{n+1}/x_1 we obtain

$$(a_{2n+1}+a_1-1)\frac{x_{n+1}}{x_1}\equiv\frac{x_{n+1}}{x_1}\left(x_1+x_{2n+1}+\sum_{j=2}^n t_jx_{2n+1}\frac{x_{n+j}}{x_j}\right).$$

Letting $y_l=x_{2n+1}x_{n+l}/x_l$, $l=1\,,\,\ldots\,,\,n$, this becomes

$$(a_{2n+1}+a_1-1)\frac{x_{n+1}}{x_1}\equiv x_{n+1}+y_1+\sum_{j=2}^n t_jy_1\frac{x_{n+j}}{x_j}.$$

Thus we are reduced to the problem of reducing $y_1 \cdots y_{l-1} x_{n+l}/x_l$ modulo $\sum_{i=1}^{2n+1} D_{a,i,t} R'$. Applying $f_S(D_a)$ with $S = \{1, \ldots, l\}$ we obtain

$$\begin{aligned}
(a_{2n+1} + a_1 + \cdots + a_l - 1) y_1 \cdots y_{l-1} \frac{x_{n+l}}{x_l} \\
&\equiv \left(x_1 + \cdots + x_l + x_{2n+1} + \sum_{j=l+1}^{n} t_j y_j \right) y_1 \cdots y_{l-1} \frac{x_{n+l}}{x_l} \\
&= y_1 \cdots y_l + \sum_{i=1}^{l} y_1 \cdots \hat{y}_i \cdots y_l x_{n+i} + \sum_{j=l+1}^{n} t_j y_1 \cdots y_l \frac{x_{n+j}}{x_j}.
\end{aligned}$$

By iteration we arrive at a representation of $1/x_1$ as a polynomial in $x_1, \ldots, x_{2n+1}, y_1, \ldots, y_n$ with coefficients in $\mathbb{Q}(a)[t]$. The number of steps is quite large since $f_{n+1}(-\epsilon_1) = -1$ and $f_S(-\epsilon_1) = -1$ for every subset S of $\{1, \ldots, n\}$ that contains 1 (thus $N_{-\epsilon_1} = 1 + 2^{n-1}$).

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